XXV. *A new Method of finding Fluents by Continuation.* By *the Rev.* Samuel Vince, *A. M. F. R. S.*

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ART. I. Put $\dot{F} = \dfrac{x^m \dot{x}}{a^n + x^n} = x^{m-n}\dot{x} - a^n x^{m-2n}\dot{x} + a^{2n} x^{m-3n}\dot{x} -$

&c. $\pm \dfrac{a^{vn} x^{m-vn} \dot{x}}{a^n + x^n}$, then $F = \dfrac{x^{m-n+1}}{m-n+1} - \dfrac{a^n x^{m-2n+1}}{m-2n+1} + \dfrac{a^{2n} x^{m-3n+1}}{m-3n+1} -$ &c.

$\pm W$, where W reprefents the fluent of the laft term. Now

$\dfrac{x^m \dot{x}}{\overline{a^n + x^n}|^r} = \dfrac{x^m \dot{x} \times \overline{a^n + x^n}|^{1-r}}{a^n + x^n}$; hence $\displaystyle\int \dfrac{x^m \dot{x}}{\overline{a^n + x^n}|^r} = \int \dfrac{x^m \dot{x} \times \overline{a^n + x^n}|^{1-r}}{a^n + x^n} = F$

$\times \overline{a^n + x^n}|^{1-r} - \displaystyle\int F \times \dfrac{\overline{1-r} \cdot n x^{n-1} \dot{x}}{\overline{a^n + x^n}|^r} = $ (by fubftituting for F its

value in the latter quantity) $F \times \overline{a^n + x^n}|^{1-r} - \dfrac{\overline{1-r} \cdot n}{m-n+1} \times$

$\displaystyle\int \dfrac{x^m \dot{x}}{\overline{a^n + x^n}|^r} + \dfrac{\overline{1-r} \cdot n a^n}{m-2n+1} \times \int \dfrac{x^{m-n} \dot{x}}{\overline{a^n + x^n}|^r} - \dfrac{\overline{1-r} \cdot n a^{2n}}{m-3n+1} \times \int \dfrac{x^{m-2n} \dot{x}}{\overline{a^n + x^n}|^r} +$ &c. $\pm$

$\displaystyle\int W \times \dfrac{\overline{1-r} \cdot n x^{n-1} \dot{x}}{\overline{a^n + x^n}|^r}$; tranfpofe $- \dfrac{\overline{1-r} \cdot n}{m-n+1} \times \int \dfrac{x^m \dot{x}}{\overline{a^n + x^n}|^r}$, and divide

both fides of the equation by $\dfrac{m-rn+1}{m-n+1}$ and we have

$\displaystyle\int \dfrac{x^m \dot{x}}{\overline{a^n + x^n}|^r} = \dfrac{m-n+1}{m-rn+1} \times F \times \overline{a^n + x^n}|^{1-r} + \dfrac{m-n+1}{m-rn+1} \times \dfrac{\overline{1-r} \cdot n a^n}{m-2n+1} \times$

$\displaystyle\int \dfrac{x^{m-n} \dot{x}}{\overline{a^n + x^n}|^r} - \dfrac{m-n+1}{m-rn+1} \times \dfrac{\overline{1-r} \cdot n a^{2n}}{m-3n+1} \times \int \dfrac{x^{m-2n} \dot{x}}{\overline{a^n + x^n}|^r} +$ &c. $\pm \dfrac{m-n+1}{m-rn+1} \times$

$\displaystyle\int$

$\int W \times \dfrac{\overline{1-r} \cdot n x^{n-1} \dot{x}}{\overline{a^n + x^n}|^r}$; now the fluent of the laſt term is

$\mp \dfrac{m-n+1}{m-rn+1} \times W \times \overline{a^n + x^n}|^{1-r} \pm \dfrac{m-n+1}{m-rn+1} \times \int \dfrac{a^{vn} x^{m-vn} \dot{x}}{\overline{a^n + x^n}|^r}$; hence by

ſubſtituting this quantity for the laſt term, it is manifeſt, that the firſt part $\mp \dfrac{m-n+1}{m-rn+1} \times W \times \overline{a^n + x^n}|^{1-r}$ will be deſtroyed

by the laſt term of $\dfrac{m-n+1}{m-rn+1} \times F \times \overline{a^n + x^n}|^{1-r}$, when we ſubſtitute

for F its value; hence if we put $M = \dfrac{x^{m-n+1}}{m-n+1} - \dfrac{a^n x^{m-2n-1}}{m-2n+1} +$

$\dfrac{a^{2n} x^{m-3n+1}}{m-3n+1} -$ &c. omitting the laſt term $\pm W$, we have

$\int \dfrac{x^m \dot{x}}{\overline{x^n + x^n}|^r} = \dfrac{m-n+1}{m-rn+1} \times M \times \overline{a^n + x^n}|^{1-r} + \dfrac{m-n+1}{m-rn+1} \times \dfrac{\overline{1-r} \cdot n a^n}{m-2n+1} \times$

$\int \dfrac{x^{n-n} \dot{x}}{\overline{a^n + x^n}|^r} - \dfrac{m-n+1}{m-rn+1} \times \dfrac{\overline{1-r} \cdot n a^{2n}}{m-3n+1} \times \int \dfrac{x^{m-2n} \dot{x}}{\overline{a^n + x^n}|^r} +$ &c. $\pm \dfrac{m-n+1}{m-rn+1} \times$

$\times a^{vn} \times \int \dfrac{x^{m-vn} \dot{x}}{\overline{a^n + x^n}|^r}$; hence, if the fluent of the laſt term be

given, we have the general law of continuation by which we may find the fluent of $\dfrac{x^m \dot{x}}{\overline{a^n + x^n}|^r}$. If the fluxion be $\dfrac{x^m \dot{x}}{\overline{a^n + x^n}|^r}$ all the

terms after the firſt will be negative, and the laſt always poſitive.

Ex. 1. Given the fluent of $\dfrac{\dot{x}}{\sqrt{1 + x^2}}$ to find the fluent of

$\dfrac{x^{25} \dot{x}}{\sqrt{1 + x^2}}$.

Here $n = 2$, $a = 1$, $m = 25$, $r = \frac{1}{2}$, $M = \dfrac{x^{2n-1}}{2n-1} - \dfrac{x^{2n-3}}{2n-3} +$ &c. to

$\pm x$,

$\pm x$, and the fluent of $\dfrac{\dot{x}}{\sqrt{1+x^2}}$ is the hyp. log. $x + \sqrt{1+x^2}$

which call Q; hence $\displaystyle\int \dfrac{x^{2s}\dot{x}}{\sqrt{1+x^2}} = \dfrac{2s-1}{2s} \times M \times \sqrt{1+x^2} + \dfrac{2s-1}{2s \cdot 2s-3}$

$\times \displaystyle\int \dfrac{x^{2s-2}\dot{x}}{\sqrt{1+x^2}} - \dfrac{2s-1}{2s \cdot 2s-5} \times \displaystyle\int \dfrac{x^{2s-4}\dot{x}}{\sqrt{1+x^2}} + \&c. \pm \dfrac{2s-1}{2s} \times Q.$

If $s=1$, $\displaystyle\int \dfrac{x^2\dot{x}}{\sqrt{1+x^2}} = \tfrac{1}{2}\sqrt{1+x^2} \times x - \tfrac{1}{2}Q = \alpha.$

$s=2$, $\displaystyle\int \dfrac{x^4\dot{x}}{\sqrt{1+x^2}} = \tfrac{3}{4}\sqrt{1+x^2} \times \dfrac{x^3}{3} - x + \tfrac{3}{4}\alpha + \tfrac{3}{4}Q = \beta.$

$s=3$, $\displaystyle\int \dfrac{x^6\dot{x}}{\sqrt{1+x^2}} = \tfrac{5}{6}\sqrt{1+x^2} \times \dfrac{x^5}{5} - \dfrac{x^3}{3} + x + \dfrac{5}{6 \cdot 3}\beta - \dfrac{5}{6}\alpha - \dfrac{5}{6}Q = \gamma.$

$s=4$, $\displaystyle\int \dfrac{x^8\dot{x}}{\sqrt{1+x^2}} = \tfrac{7}{8}\sqrt{1+x^2} \times \dfrac{x^7}{7} - \dfrac{x^5}{5} + \dfrac{x^3}{3} - x + \dfrac{7}{8 \cdot 5}\gamma - \dfrac{7}{8 \cdot 3}$

$\beta + \dfrac{7}{8}\alpha + \dfrac{7}{8}Q.$

&c.   &c.   &c.

Ex. 2. To find the fluent of $x^{\frac{s}{2}}\dot{x}\sqrt{2+x}$, given the fluent of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$, and s an odd number.

Here $a=2$, $n=1$, $r=-\tfrac{1}{2}$, $\dfrac{s}{2}=m$, $m-vn=-\tfrac{1}{2}$, or $\dfrac{s}{2}-v$

$= -\tfrac{1}{2}$; $\therefore v=\dfrac{s+1}{2}$, $M = \dfrac{2x^{\frac{s}{2}}}{s} - \dfrac{4x^{\frac{s}{2}-1}}{s-2} + \dfrac{8x^{\frac{s}{2}-2}}{s-4} - \&c.$ and the

fluent (Q) of $x^{-\frac{1}{2}}\dot{x}\sqrt{2+x}$ is $\pi + \sqrt{2x+x^2}$, where $\pi = $ hyp.

log. $1 + x + \sqrt{2x+x^2}$; hence $\displaystyle\int x^{\frac{s}{2}}\dot{x}\sqrt{2+x} = \dfrac{s}{s+3} \times \overline{2+x|}^{\frac{3}{2}} \times$

$M + \dfrac{s}{s+3} \times \dfrac{b}{s-2} \times \displaystyle\int x^{\frac{s}{2}-1}\dot{x}\sqrt{2+x} - \dfrac{s}{s+3} \times \dfrac{12}{s-4} \times \displaystyle\int x^{\frac{s}{2}-2}\dot{x}\sqrt{2+x}$

$+ \dfrac{s}{s+3} \times \dfrac{24}{s-6} \times \displaystyle\int x^{\frac{s}{2}-3}\dot{x}\sqrt{2+x}$ &c. $\pm \dfrac{s}{s+3} \times 2^{\frac{s+1}{2}} \times \displaystyle\int x^{-\frac{1}{2}}\dot{x}$

$\sqrt{2+x}.$

If

If $s = 1$, $\int x^{\frac{1}{2}}\dot{x} \sqrt{\overline{2+x}} = \frac{1}{4} \times \overline{2+x}^{\frac{3}{2}} \times 2x^{\frac{1}{2}} - \frac{1}{2}Q = \alpha.$

$s = 3$, $\int x^{\frac{3}{2}}\dot{x} \sqrt{\overline{2+x}} = \frac{1}{2} \times \overline{2+x}^{\frac{3}{2}} \times \frac{2x^{\frac{3}{2}}}{3} - 4x^{\frac{1}{2}} + 3\alpha + 2Q = \beta.$

$s = 5$, $\int x^{\frac{5}{2}}\dot{x} \sqrt{\overline{2+x}} = \frac{5}{8} \times \overline{2+x}^{\frac{3}{2}} \times \frac{2x^{\frac{5}{2}}}{5} - \frac{4x^{\frac{3}{2}}}{3} + 8x^{\frac{1}{2}} + \frac{5}{4}\beta -$

$\frac{15}{2}\alpha - 5Q = \gamma.$

$s = 7$, $\int x^{\frac{7}{2}}\dot{x} \sqrt{\overline{2+x}} = \frac{7}{10} \times \overline{2+x}^{\frac{3}{2}} \times \frac{2x^{\frac{7}{2}}}{7} - \frac{4x^{\frac{5}{2}}}{5} + \frac{8x^{\frac{3}{2}}}{3} - 16x^{\frac{1}{2}} +$

$\frac{21}{25}\gamma - \frac{14}{5}\beta + \frac{84}{5}\alpha + \frac{56}{5}Q.$

&c. &c. &c.

II. Let $\dot{F} \dfrac{x^n \dot{x}}{a + bx^m + x^{2m}} = x^{n-2m}\dot{x} - Px^{n-3m}\dot{x} + Qx^{n-4m}\dot{x} - $ &c. $\pm$

$\dfrac{Vx^{n-sm}\dot{x}}{a + bx^m + x^{2m}} \pm \dfrac{Wx^{\overline{n-s+1}\cdot m}\dot{x}}{a + bx^m + x^{2m}}$, then $F = \dfrac{x^{n-2m+1}}{n-2m+1} - \dfrac{Px^{n-3m-1}}{n-3m+1} + \dfrac{Qx^{n-4m+1}}{n-4m+1}$

$-$ &c. $\pm T \pm U$, where T and U are put for the fluents of the two laſt terms, and P, Q, &c. for the co-efficients ariſing from

the diviſion. Now, $\displaystyle\int \dfrac{x^n \dot{x}}{\sqrt{a + bx^m + x^{2m}}} = \int \dfrac{x^n \dot{x} \times \overline{a + bx^m + x^{2m}}^{\frac{1}{2}-r}}{a + bx^m + x^{2m}} =$

$F \times \overline{a + bx^m + x^{2m}}^{\frac{1}{2}-r} - \displaystyle\int F + \dfrac{\overline{\frac{1}{2}-r} \cdot mbx^{m-1}\dot{x} + \overline{\frac{1}{2}-r} \, 2mx^{2m-1}\dot{x}}{\overline{a + bx^m + x^{2m}}^{r}} = $ (by

fubſtituting for F its value in the latter quantity, and putting A, B, C, &c for the co-efficients which ariſe in conſequence

thereof) $F \times \overline{a + bx^m + x^{2m}}^{\frac{1}{2}-r} - A \times \displaystyle\int \dfrac{x^n \dot{x}}{\overline{a + bx^m + x^{2m}}^{r}} + B \times$

$\displaystyle\int \dfrac{x^{n-m}\dot{x}}{\overline{a + bx^m + x^{2m}}^{r}} - C \times \displaystyle\int \dfrac{x^{n-2m}\dot{x}}{\overline{a + bx^m + x^{2m}}^{r}} + $ &c.

$$\mp \int T \times \frac{\overline{1-r} \cdot mbx^{m-1}\dot{x} + \overline{1-r} \cdot 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$$

$$\mp \int U \times \frac{\overline{1-r} \cdot mbx^{m-1}\dot{x} + \overline{1-r} \cdot 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r};\text{ hence by tranfpofition}$$

and divifion we have $\displaystyle\int \frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} = \frac{1}{1+A} \times F \times \overline{a+bx^m+x^{2m}}|^{1-r}$

$$+ \frac{B}{1+A} \times \int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} - \frac{C}{1+A} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} + \&c.$$

$$\mp \int \frac{T}{1+A} \times \frac{\overline{1-r} \cdot mbx^{m-1}\dot{x} + \overline{1-r} \cdot 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}$$

$$\mp \int \frac{U}{1+A} \times \frac{\overline{1-r} \cdot mbx^{m-1}\dot{x} + \overline{1-r} \cdot 2mx^{2m-1}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r}. \text{ Now the fluents}$$

of thefe two laft terms are $\mp\dfrac{T}{1+A} \times \overline{a+bx^m+x^{2m}}|^{1-r} \pm \dfrac{V}{1+A}$

$\displaystyle\int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r},$ and $\mp\dfrac{U}{1+A} \times \overline{a+bx^m+x^{2m}}|^{1-r} \pm \dfrac{W}{1+A}\displaystyle\int \frac{x^{n-s+1} \cdot m\dot{x}}{\overline{a+bx^m+x^{2m}}|^r},$

refpectively ; hence, by fubftituting thefe for the laft term, it is manifeft that $\mp \dfrac{T}{1+A} \times \overline{a+bx^m+x^{2m}}|^{1-r}$ and $\mp \dfrac{U}{1+A}$

$\times \overline{a+bx^m+x^{2m}}|^{1-r}$ will be deftroyed by the two laft terms of

$\dfrac{1}{1+A} \times F \times \overline{a+bx^m+x^{2m}}|^{1-r}$ when we fubftitute for F its value ;

hence, if we put $M = \dfrac{x^{n-2m-1}}{n-2m+1} - \dfrac{Px^{n-3m+1}}{n-3m+1} + \dfrac{Qx^{n-4m+1}}{n-4m+1} - \&c.$

omitting the two laft terms $\pm$ T and $\pm$ U, we fhall have

$$\int \frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} = \frac{1}{1+A} \times M \times \overline{a+bx^m+x^{2m}}|^{1-r} + \frac{B}{1+A} \times$$

$$\int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} - \frac{C}{1+A} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+x^{2m}}|^r} + \&c. \pm \frac{V}{1+A} \times$$

$$\int$$

$$\int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+x^{2m}}\,|^r} \doteq \frac{W}{1+A} \times \frac{x^{\,n-s+1\,.\,m}\dot{x}}{\overline{a+bx^m+x^{2m}}\,|^r} \; ;$$ hence if the two laſt

fluents be given, we have the general law of continuation up to

$$\frac{x^n\dot{x}}{\overline{a+bx^m+x^{2m}}\,|^r}$$ in the ſame manner as before.

III. In general, if we proceed as in the two laſt articles, we

ſhall find $$\int \frac{x^n\dot{x}}{\overline{a+bx^m+\&c.\; x^{tm}}\,|^r} = \frac{M}{P} \times \overline{a+bx^m+\&c.\; x^{tm}}\,|^{1-r} + \frac{A}{P}$$

$$\times \int \frac{x^{n-m}\dot{x}}{\overline{a+bx^m+\&c.\; x^{tm}}\,|^r} + \frac{B}{P} \times \int \frac{x^{n-2m}\dot{x}}{\overline{a+bx^m+\&c.\; x^{tm}}\,|^r} + \&c. \doteq \frac{T}{P}$$

$$\times \int \frac{x^{n-sm}\dot{x}}{\overline{a+bx^m+\&c.\; x^{tm}}\,|^r} \doteq \frac{V}{P} \times \int \frac{x^{\,n-s+1\,.\,m}\dot{x}}{\overline{a+bx^m+\&c.\; x^{tn}}\,|^r} \doteq \&c.$$ where the

number of theſe laſt terms is t, and $M = \dfrac{x^{n-tm+1}}{n-tm+1} - \dfrac{Qx^{\,n-t+1\,.\,m-1}}{n-t+1\,.\,m+1}$

$+$ &c. omitting, as before, the terms at the end ariſing from
the remainders. Hence if the laſt t fluents be given, we can
by continuation find the required fluent.

Becauſe the diviſion of $\dfrac{x^n\dot{x}}{a+bx^{m}+\&c.\; x^{tm}}$ may be expreſſed by an

aſcending ſeries $x^n\dot{x} - Qx^{n+m}\dot{x} + Rx^{n+2m}\dot{x} - \&c.$ it is manifeſt, that
by the ſame method we may continue the fluents downwards
as well as upwards.

IV. Let $\dot{F} = \dfrac{x^n\dot{x}}{1-x} = -x^{n-1}\dot{x} - x^{n-2}\dot{x} - x^{n-3}\dot{x} - \&c. - x^{n-r}\dot{x} +$

$\dfrac{x^{n-r}\dot{x}}{1-x}$, then $F = -\dfrac{x^n}{n} - \dfrac{x^{n-1}}{n-1} - \dfrac{x^{n-2}}{n-2} - \&c. - \dfrac{x^{n-r+1}}{n-r+1} + W$, where

W is the fluent of the laſt term. Now $\dfrac{x^n\dot{x}}{\sqrt{1-x^2}} = \dfrac{x^n\dot{x}}{1-x} \times \sqrt{\dfrac{1-x}{1+x}}$,

hence

hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = \int \frac{x^n \dot{x}}{1-x} \times \sqrt{\frac{1-x}{1+x}} = F \times \sqrt{\frac{1-x}{1+x}} + \int F \times$

$\frac{\dot{x}}{\sqrt{1+x^2} \times 1+x} = F \times \sqrt{\frac{1-x}{1+x}} - \int \frac{x^n \dot{x}}{n\sqrt{1-x^2} \times 1+x} - \int \frac{x^{n-1} \dot{x}}{n-1 . \sqrt{1-x^2} \times 1+x}$

$- \int \frac{x^{n-2} \dot{x}}{n-2 . \sqrt{1-x^2} \times 1+x} - \&c. - \int \frac{x^{n-r+1} \dot{x}}{n-r+1 . \sqrt{1-x^2} \times 1+x} +$

$\int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times 1+x}$. But

$\frac{x^n \dot{x}}{1-x^2 \times 1+x} = \frac{x^{n-1} \dot{x}}{n\sqrt{1-x^2}} - \frac{x^{n-2} \dot{x}}{n\sqrt{1-x^2}} + \frac{x^{n-3} \dot{x}}{n\sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r} \dot{x}}{n\sqrt{1-x^2}} \pm \frac{x^{n-r} \dot{x}}{n\sqrt{1-x^2} \times 1+x}$

$\frac{x^{n-1} \dot{x}}{\sqrt{1-x^2} \times 1+x} = \frac{x^{n-2} \dot{x}}{n-1 . \sqrt{1-x^2}} - \frac{x^{n-3} \dot{x}}{n-1 . \sqrt{1-x^2}} + \&c. \pm \frac{x^{n-r} \dot{x}}{n-1 . \sqrt{1-x^2}} \mp \frac{x^{n-r} \dot{x}}{n-1 . \sqrt{1-x^2} \times 1+x}$

$\frac{x^{n-2} \dot{x}}{\sqrt{1-x^2} \times 1+x} = \frac{x^{n-3} \dot{x}}{n-2 . \sqrt{1-x^2}} - \&c. \mp \frac{x^{n-r} \dot{x}}{n-2 . \sqrt{1-x^2}} \pm \frac{x^{n-r} \dot{x}}{n-2 . \sqrt{1-x^2} \times 1+x}$

&c.   &c.   &c.

$\frac{x^{n-r+1} \dot{x}}{-1 . \sqrt{1-x^2} \times 1+x} = \frac{x^{n-r} \dot{x}}{n-r+1 . \sqrt{1+x^2}} - \frac{x^{n-r} \dot{x}}{n-r+1 . \sqrt{1-x^2} \times 1+x}$.

Hence $\int \frac{x^n \dot{x}}{\sqrt{1-x^2}} = F \times \sqrt{\frac{1-x}{1+x}} - \frac{1}{n} \times \int \frac{x^{n-1} \dot{x}}{\sqrt{1-x^2}} + \overline{\frac{1}{n} - \frac{1}{n-1}} \times$

$\int \frac{x^{n-2} \dot{x}}{\sqrt{1-x^2}} - \overline{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}} \times \int \frac{x^{n-3} \dot{x}}{\sqrt{1-x}} + \&c. \pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c.$

$- \overline{\frac{1}{n-r+1}} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}} \mp \frac{1}{n} \pm \frac{1}{n-1} \mp \&c + \frac{1}{n-r+1} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x} \times 1+x}$

$+ \int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times 1+x}$. Now $\int W \times \frac{\dot{x}}{\sqrt{1-x^2} \times 1+x} = - W \times$

$\overline{\frac{1-x}{1+x}}|^{\frac{1}{2}} + \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}}$; alfo $\int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2} \times 1+x} = - x^{n-r} \times \sqrt{\frac{1-x}{1+x}} +$

$\overline{n-r} \times \int \frac{x^{n-r-1} \dot{x}}{\sqrt{1-x^2}} - \overline{n-r} \times \int \frac{x^{n-r} \dot{x}}{\sqrt{1-x^2}}$; hence, by fubftituting

thefe quantities in the two laft terms, it is manifeft that

$- W \times \overline{\frac{1-x}{1-x}}|^{\frac{1}{2}}$ will be deftroyed by the laft term of

F ×

$F \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}}$ when we fubftitute for F its value; therefore, if we

put $M = -\frac{x^n}{n} - \frac{x^{n-1}}{n-1} - \frac{x^{n-2}}{n-2} - \&c. - \frac{x^{n-r+1}}{n-r+1}$, we fhall have

$$\int \frac{x^n \dot{x}}{\sqrt{1+2}} = M \pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1} \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}} - \frac{1}{n} \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}}$$

$$+ \overline{\frac{1}{n} - \frac{1}{n-1}} \times \int \frac{x^{n-2}\dot{x}}{\sqrt{1-x^2}} - \overline{\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}} \times \int \frac{x^{n-3}\dot{x}}{\sqrt{1-x^2}} + \&c.$$

$$\pm \frac{1}{n} \mp \frac{1}{n-1} \pm \&c. - \frac{1}{n-r+1} \times \overline{n-r+1} + 1 \times \int \frac{x^{n-r}\dot{x}}{\sqrt{1-x^2}} \mp$$

$\frac{1}{n} \pm \frac{1}{n-1} \mp \&c. + \frac{1}{n-r+1} \times \overline{n-r} \times \int \frac{x^{n-r-1}\dot{x}}{\sqrt{1-x^2}}$. Hence, if the

two laft fluents be given, we have the general law of conti‑

nuation up to the fluent of $\frac{x^n \dot{x}}{\sqrt{1-x^2}}$, where the index of x with‑

out the vinculum increafes by unity each time. And in the

fame manner we may (by increafing the index of x without by

m) find the fluent of $\frac{x^n \dot{x}}{\sqrt{a^{2m}-x^{2m}}}$ if we have given the fluents of

$\frac{x^{n-rm}\dot{x}}{\sqrt{a^{2n}-x^{2m}}}$ and $\frac{x^{n-\overline{r}+1\cdot m}\dot{x}}{\sqrt{a^{2m}-x^{2m}}}$. Thus we have a general law of con‑

tinuation, where the index of x without is increafed by half
the index under the vinculum.

V. Affume $\dot{F} = \frac{x^n\dot{x}}{x^m-b} = x^{n-m}\dot{x} + bx^{n-2m}\dot{x} + b^2x^{n-3m}\dot{x} + \&c. +$

$\frac{b^r x^{n-rm}\dot{x}}{x^m-b}$, then $F = \frac{x^{n-m+1}}{n-m+1} + \frac{bx^{n-2m+1}}{n-2m+1} + \frac{b^2 x^{n-3m+1}}{n+3m+1} + \&c. + W$,

where W is put for the fluent of the laft term. Now

$$\int x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} = \int \frac{x^n}{x^m-b} \times \sqrt{x^m-a \times x^m-b} = F \times \sqrt{x^m-a \times x^m+b}$$

$$-\int F \times \frac{2mx^{2m-1}\dot{x}-\overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{x^m-a\times x^m-b}} = \text{(by fubftituting for F its}$$

value in the latter quantity, and putting A, B, C, &c. for the co-efficients which arife therefrom) $F \times \sqrt{x^m-a\times x^m-b}\,-$

$$A\int \frac{x^n\dot{x}}{\sqrt{x^m-a\times x^m-b}} - B\int \frac{x^{n-m}\dot{x}}{\sqrt{x^m-a\times x^m-b}} - C\int \frac{x^{n-2m}\dot{x}}{\sqrt{x^m-a\times x^m-b}}\,-$$

$$\&c. - \int W \times \frac{2mx^{2m-1}\dot{x}-\overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{x^m-a\times x^m-b}}. \quad \text{Now} \quad \frac{x^n\dot{x}}{\sqrt{x^m-a\times x^m-b}} =$$

$$\frac{x^n\dot{x}}{x^m-a}\sqrt{\frac{x^m-a}{x^m-b}}, \quad \frac{x^{n-m}\dot{x}}{\sqrt{x^m-a\times x^m-b}} = \frac{x^{n-m}\dot{x}}{x^m-a}\sqrt{\frac{x^m-a}{x^m-b}}, \quad \frac{x^{n-2m}\dot{x}}{\sqrt{x^m-a\times x^m-b}} = \frac{x^{n-2m}\dot{x}}{x^m-a}$$

$$\sqrt{\frac{x^m-a}{x^m-b}}, \text{ &c. But}$$

$$\frac{x^n\dot{x}}{x^m-a} = x^{n-m}\dot{x} + ax^{n-2m}\dot{x} + a^2x^{n-3m}\dot{x} + \&c. + a^{r-1}x^{n-rm}\dot{x} + \frac{a^rx^{n-rm}\dot{x}}{x^m-a}$$

$$\frac{x^{n-m}\dot{x}}{x^m-a} = \qquad x^{n-2m}\dot{x} + ax^{n-3m}\dot{x} + \&c. + a^{r-2}x^{n-rm}\dot{x} + \frac{a^{r-1}x^{n-rm}\dot{x}}{x^m-a}$$

$$\frac{x^{n-2m}\dot{x}}{x^m-a} = \qquad x^{n-3m}\dot{x} + \&c. + a^{r-3}x^{n-rm}\dot{x} + \frac{a^{r-2}x^{n-rm}\dot{x}}{x^m-a}$$

$$\&c. \qquad\qquad \&c. \qquad\qquad \&c.$$

Hence, by fubftitution, $\int x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} = F \times \sqrt{x^m-a\times x^m-b}$

$$-A\int x^{n-m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa+B}\times\int x^{n-2m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^2+Ba+C}$$

$$\times\int x^{n-3m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \&c. - \overline{Aa^{r-1} + Ba^{r-2} + Ca^{r-3} + \&c.} \times]$$

$$\times\int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^r + Ba^{r-1} + Ca^{r-2} + \&c.} \times\int \frac{x^{n-rm}\dot{x}}{\sqrt{x^m-a+x^m-b}}$$

$$-\int W \times \frac{2mx^{2m-1}\dot{x}-\overline{a+b}\,.\,mx^{m-1}\dot{x}}{2\sqrt{x^m-a\times x^m-b}}. \quad \text{But}$$

$$-\int W \times \frac{2mx^{2m-1}\dot{x}-\overline{+b}\cdot mx^{m-1}\dot{x}}{2\sqrt{x^m-a\times x^m-b}} \text{ is } -W \times \sqrt{x^m-a\times x^m-b}$$

$$+b^r \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}}\,; \text{ hence, by fubftituting this for the laft}$$

term, it is manifeft, that $-W \times \sqrt{x^m-a\times x^m-b}$ will be de-

ftroyed by the laft term of $F \times \sqrt{x^m-a\times x^m-b}$ when we fub-

ftitute for F its value; therefore, if we put $M = \dfrac{x^{n-m+1}}{n-m+1} +$

$\dfrac{bx^{n-2m+1}}{n-2m+1} + \&c. + \dfrac{b^{r-1}x^{n-rm+1}}{n-rm+1}$, we have $\displaystyle\int x^n\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} = M \times$

$$\sqrt{x^m-a\times x^m-b} - A\int x^{n-m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa+B} \times \int x^{n-2m}\dot{x}$$

$$\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^2+Ba+C} \times \int x^{n-3m}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \&c. -$$

$$\overline{Aa^{r-1}+Ba^{r-2}+\&c.-b^r} \times \int x^{n-rm}\dot{x}\sqrt{\frac{x^m-a}{x^m-b}} - \overline{Aa^r+Ba + Ca^{r-2}}$$

$$+\&c. \times \int \frac{x^{n-rm}\dot{x}}{\sqrt{x^m-a\times x^m-b}}. \text{ Hence if the laft two fluents be}$$

given, we have the general law of continuation up to the

fluent of $x^n\dot{x}\sqrt{\dfrac{x^m-a}{x^m-b}}$.

The utility of finding fluents by continuation was manifeft
to Sir Isaac Newton, who firft propofed it; and fince his
time fome of the moft eminent mathematicians have employed
much of their attention upon it. The method which I have
inveftigated and exemplified in this Paper I offer as being en-
tirely new; and at the fame time it not only exhibits, at once,
the general law up to the required fluent, but alfo appears, from
fome of the inftances here given, to be more extenfive and con-
venient in its application than any method hitherto offered.

<div align="right">The</div>

The general refolution of the given fluxion into a feries of fluxions of the fame kind, where the index of the unknown quantity without the vinculum keeps decreafing or increafing either by the index under or by half the index, has not, that I know of, before been given; which furnifhes us at once not only with a very eafy method of *continuing* fluents, but alfo points out a very fimple method of inveftigating the fluent of the given fluxion *without* continuation. For if $\int \dot{A} = p + b \int \dot{B} + c \int \dot{C} + d \int \dot{D} + $&c. $\int \dot{B} = p' + c' \int \dot{C} + d' \int \dot{D} + $ &c. $\int \dot{C} = p'' + d'' \int \dot{D} + $&c. &c. &c. then if for $\int \dot{B}, \int \dot{C},$ &c. &c. we fubftitute their refpective values, we fhall get a general feries for $\int \dot{A}$ without continuation.   The extent of any new method is, at firft, feldom obvious; and how far that which is here propofed may be fuccefsfully employed in other cafes will beft appear from its application.   Different methods will always be found to have their ufes in particular cafes; for where one be- comes impracticable another will often be found to fucceed; and I hope that which is here offered will contribute fome- thing towards facilitating the inveftigation of fluents.